[1] [2, 3] [4] [5] [6, 7, 8]) [6] [9]

1 Lie Groups and the Noether Theorem

In this section we are going to deal with the geometric properties of the prior $p(\nabla\phi)$, specifically the properties of the set of maxima The idea is to introduce a methodology for designing $p(\nabla\phi)$ whose set of maxima A respects certain geometric constraints related to a group of smooth transistions $\mathbb{G}$. This methodology will us deduce explicit constraints on $p(\nabla\phi)$ from expected geometric properties of A. The starting point is the idea that A can be seen as being generated by the group $\mathbb{G}$

$$A = \{\phi^\star \,|\phi^\star = g \circ \phi_0^\star \quad g \in \mathbb{G}\} \tag{1}$$

for some maximum $\phi_0^\star$. Since $\mathbb{G}$ is assumed to be a Lie group, that is, it is closed under the group product, the choice of $\phi_0^\star$ is arbitrary. This way $\mathbb{G}$ defines the topology of the set A, and as we will show, leads to a differential equation on $p(\nabla\phi)$.

1.1 Lie Groups

Lie groups are groups of objects which are smooth functions on a manifold at the same time being compatible with the group multiplication. For an n dimensional Lie group $\mathbb{G}$ over a domain Ω the map

$$\mathbb{G} \times \mathbb{G} \longmapsto \mathbb{G} : (x, y) \rightarrow x \cdot y^{-1}$$

is smooth in both x and y. One of the remarkable properties of Lie group theory is that for every Lie group $\mathbb{G}$ there exists a unique structure called a Lie algebra $\mathcal{G}$. The Lie algebra is assotiated with the tangent space of $\mathbb{G}$, $\mathbb{TG}$. We introduce the Lie algebra by defining a one parameter Lie group $\gamma(t)$

$$\gamma : \mathbb{R} \rightarrow \mathbb{G}_\gamma \in \mathbb{G} \tag{2}$$

$$\gamma(0) = e \tag{3}$$

$$\left.\frac{d}{dt}\gamma(t)\right|_{t=0} = X \in \mathcal{G} \tag{4}$$

The path γ is not unique, there are infinit many paths which have the same tangential vector X at $t = 0$. This defines an equivalence relation: the paths γ_1 and γ_2 are equivalent, $\gamma_1 \sim \gamma_2$ if

$$\left.\frac{d}{dt}\gamma_1(t)\right|_{t=0} = \left.\frac{d}{dt}\gamma_2(t)\right|_{t=0} \tag{5}$$

By Eq. eq. (5) we can group the paths γ into equivalence classes $[\gamma]$ For the vector X in eq. (4) we can find a descriptive formulation in form of a vector field $\omega(\mathbf{z}_0), \mathbf{z}_0 \in \mathcal{M}$ where the space $\mathcal{M}$ is a smooth manifold upon which $\mathbb{G}$ acts. The vector field X induces a path $\Gamma^X(\mathbf{z}_0, s)$ with the properties

$$\frac{d}{dt}\mathbf{\Gamma}^X(\mathbf{z}_0, t) = X\left(\mathbf{\Gamma}^X(\mathbf{z}_0, t)\right) \tag{6}$$

$$\left.\mathbf{\Gamma}^X(\mathbf{z}_0, t)\right|_{t=0} = \mathbf{z}_0; \tag{7}$$

The coordinates of X relative to the space $\mathcal{M}$ can be computed when we look at the space of smooth functions with support on $\mathcal{M}$, $\mathcal{F}(\mathcal{M})$. The action of X on $\mathcal{F}(\mathcal{M})$ can be computed by evaluating $F \in \mathcal{F}(\mathcal{M})$ on the integral curve $\Gamma^X(\mathbf{z}_0, s)$ and taking the derivative

$$\frac{d}{ds} F\left(\Gamma^X(\mathbf{z}_0, s)\right)\Big|_{s=0} = XF\left(\Gamma^X(\mathbf{z}_0, s)\right)\big|_{t=0} \tag{8}$$

If F is one of the coordinate functions z_i then the vector field X has the coefficients $\omega_i(\mathbf{z}_0)$ relative to the basis in $\mathcal{M}$

$$X \cdot z_i|_{\mathbf{z}=\mathbf{z}_0} = \omega_i(\mathbf{z}_0) \tag{9}$$

The set of differential operators $\{\partial_{z_i}\}$ can be interpreted as a basis of the Lie algebra and the vector field $\omega(\mathbf{z}_0)$ is the coordinate representation of X at the point $\mathbf{z}_0 \in M$. We would like build upon the discussion about the one dimensional Lie Group $\mathbb{G}_\gamma$ and generalize it higher dimensionnal groups. An m-dimensional Lie Group $\mathbb{G}$ is a set of elements which are parameterized by m parameters

$$g_{a_1 \ldots a_m} \in \mathbb{G} \tag{10}$$

The parameters a_l define axis in the set $\mathbb{G}$ which themselves are one parameter subgroups. For each parameter a_l there exists by extension of eq. (4) a vector field X_l

$$\frac{d}{da_l} g_{a_1 \ldots a_m}\Big|_{a_1 \ldots a_m = 0} = X_l \tag{11}$$

Just like in eq. (8) the vectorfields X_l each have a coordinate representation relative to the space $\mathcal{M}$

$$X_l = \sum_i \xi_i^l(\mathbf{z}_0) \cdot \partial_{z_i} \tag{12}$$

The vector fields X_l constitute a basis for the Lie Algebra $\mathcal{G}$. All one parameter subgroups γ (see eq. (4)) can be represented as paths in the parameterspace of $\mathbb{G}$

$$\gamma(t) = g_{a_1(t) \ldots a_m(t)} \tag{13}$$

and the vectorfield X from eq. (4) is computed from the derivatives of the parameters

$$\frac{d}{dt}\gamma(t)\Big|_{t=0} = \sum_l \frac{d}{dt} a_l \cdot \frac{d}{da_l} g_{a_1 \ldots a_m}\Big|_{t=0} = \sum_l \alpha_l X_l \tag{14}$$

When we combine eq. (14) and eq. (12) we get a coordinate expression for the coeffitient vectorfield ω in the basis of the Lie Algebra

$$\omega^i(\mathbf{z}) = \sum_l \alpha_l \cdot \xi^i(\mathbf{z}) \tag{15}$$

The Lie algebra is connected to the infinitesimal Lie group $\mathbb{U}_r \subset \mathbb{G}$ via

$$g_{\mathbb{U}} = \mathbb{1} + X \in \mathbb{U}_r \tag{16}$$

For the rest of this work we will focus on the space $\mathcal{M} = \Omega \times J^k(\mathcal{C}^\infty(\Omega))$. The space $\Omega \subset \mathbb{R}^n$ is an open subset and $J^k(\mathcal{C}^\infty(\Omega))$ is the set of smooth differentiable functions with compact support in Ω and their derivatives up to order k. The points $\mathbf{z} \in \mathcal{M}$ are vectors of the *independent* variables $\mathbf{x}$, the *dependent* variable $\phi(\mathbf{x})$ and its derivatives $\phi_{,K}$ where K is a multiindex

$$\mathbf{z} = (\mathbf{x}, \phi(\mathbf{x}), \phi_{,K}(\mathbf{x})) \tag{17}$$

For this work we will focus only on first order derivatives, $k = 1$ The vector field X then has the form

$$X = \omega_\phi(\mathbf{z}_0)\,\partial_\phi + \sum_{i=1}^{n} \omega_i(\mathbf{z}_0)\,\partial_{x_i} \tag{18}$$

Under the action of $g_{\mathbb{U}}$ points in $\mathcal{C}^\infty(\Omega) \times \Omega$ transform like

$$\begin{pmatrix} \mathbf{x}' \\ \phi'(\mathbf{x}') \\ \phi'_{,j}(\mathbf{x}') \end{pmatrix} = \begin{pmatrix} \mathbf{x} + \omega(\mathbf{x}) \\ \phi(\mathbf{x}) + \omega_\phi(\mathbf{x}) + \sum_i \omega_i(\mathbf{x})\,\partial_{x_i}\phi(\mathbf{x}) \\ \phi_{,j}(\mathbf{x}) + \omega_{\phi,j}(\mathbf{x}) - \sum_i \partial_{x_i}\phi\,\partial_{x_i}\omega_j(\mathbf{x}) \end{pmatrix} \tag{19}$$

1.2 Noether's Theorem

1.2.1 Motivation

In this section we will focus on geometric properties of the prior in our model. We will assume the prior $P(\nabla\phi)$ which is only depends first order gradients of the field $\phi(\mathbf{x})$. In general, priors have infinit sets of maxima $\phi^\star$. For instance the maximas of the prior $P_{L_2}(\nabla u) = -\ln \|\nabla u\|_{L_2}$ in eq. (42) form the set of constant fields in the domain Ω

$$A = \{u^\star \,|u^\star(\mathbf{x}) = \mathit{const}\,\mathbf{x} \in \Omega\} \tag{20}$$

It is trivial to see that there exists a one parameter Lie group of elements g_c which take A into its self

$$g_c :\ A \to A, \quad u^\star \to u^\star + c, \quad c = \text{const in } \Omega \tag{21}$$

and that $P_{L_2}(\nabla u)$ is invariant under the action of $\{g_c\}$. Since c is constant in Ω, $\{g_c\}$ is the only Lie group under which $P_{L_2}(\nabla u)$ is invariant. Knowledge of the solution set A allows us to parameterize the solutions of the complete model in eq. (42) by

$$u(\mathbf{x}) = m + h(\mathbf{x}) \quad m = \left\|u^0\right\|_{L_2} \tag{22}$$

It is known that the global mean value of the solution $u^\star$ in the model in eq. (42) is equal to that of the data u^0 Thus the parameterization allows any gradient algorithm to converge faster to the solution $u^\star(\mathbf{x})$ given the initial guess

$$u_{init}(\mathbf{x}) = m, \quad m = \left\|u^0\right\|_{L_2} \tag{23}$$

The solution space A of $P_{L_2}(\nabla\phi)$ is too trivial for most real applications, since $P_{L_2}(\nabla\phi)$ penalizes any other solution $u^\star \notin A$ which contains struckture. We want to assume that a more general prior $P(\nabla\phi)$ which has maxima $\phi^\star(\mathbf{x}) \neq \mathit{const}$. The level-sets of each field $\phi^\star(\mathbf{x})$ are taken to be the integral curves

$\mathbf{\Gamma}^X(s)$ corresponding to a vectorfield X of some Lie algebra, $X \in \mathcal{G}$. The space of the maxima of $P(\nabla\phi)$ is then entirely determined by the Lie algebra $\mathcal{G}$

$$A_{\mathbb{G}} = \left\{ \phi^\star \left| \frac{d}{ds}\phi^\star\left(\mathbf{\Gamma^X}(\mathbf{s})\right) = 0 \forall X \in \mathcal{G} \right. \right\} \tag{24}$$

The action of the Lie group on elements of its own Lie algebra preserves the algebra

$$g \cdot X \cdot g^{-1} \in \mathcal{G} \quad \forall X \in \mathcal{G},\, g \in \mathbb{G} \tag{25}$$

and by construnction the set maxima in eq. (24) it is apparent that $\mathbb{G}$ takes $A_{\mathbb{G}}$ onto itself

$$g \cdot A_{\mathbb{G}} = A_{\mathbb{G}} \quad \forall g \in \mathbb{G} \tag{26}$$

Since we have caracterized the prior $P(\nabla\phi)$ by the set of its maxima in eq. (24) which is explained by the algebra of the group $\mathbb{G}$ we denote $P(\nabla\phi)$ as being conditioned on $\mathbb{G}$, $P(\nabla\phi\,|g \in \mathbb{G})$. But due to the invariance the maxima set in eq. (24) under the action of $\mathbb{G}$, $P(\nabla\phi)$ is invariant under $\mathbb{G}$

$$P(\nabla\phi\,|g \in \mathbb{G}) = const \quad \text{w.r.t}\, g \in \mathbb{G} \tag{27}$$

This property we call *Conditional Invariance*. The most important aspect of the above discussion is that given geometrical assumptions on the solutions $\phi^\star$ in terms of $\mathbb{G}$ and $A_{\mathbb{G}}$ the condition in eq. (27) must be fullfilled, and thus it serves as a guidance in the design of the prior $P(\nabla\phi\,|g \in \mathbb{G})$.

1.2.2 Noethers First Theorem

We are now going to make eq. (27) more precise by considering the negative log-prior energy

$$I = -\ln P(\nabla\phi) = \int_\Omega \mathcal{E}(x, \nabla\phi)\, dx \tag{28}$$

we are interested in the action of $\mathbb{G}$ (see eq. (19)]. The energy in eq. (28) is said to preserved under the Lie group $\mathbb{G}$ if the following relation holds

$$I' = \int_\Omega \mathcal{E}'(x', \nabla\phi')\, dx' = \int_\Omega \left\{ \mathcal{E}(x, \nabla\phi) + \partial_i \delta Q^i \right\} dx \tag{29}$$

where the vectorfield δQ^i is some arbitrary smooth function. If eq. (29) holds then the resulting Euler-Lagrange equations $[I]$ remain unchanged and thus $\mathbb{G}$ is a symmetry of the Euler-Lagrange equations. In [2, 3] it was reasoned that the knowledge of the symmetries of the Euler-lagrange equations $[I]$ can be used to make assumptions on the form of the solutions $\phi^\star$ and thus narrow down the solution space. To be more precise, the first Noether Theorem states that if the energy integral in eq. (29) is preserved under the transformations eq. (19) then the Euler-lagrange equations must fullfill

$$[I]\,\omega_\phi = \partial_\mu (W^\mu - \delta Q^\mu) \tag{30}$$

where

$$[I] = \frac{\delta I}{\delta\phi} - \frac{d}{dx^\nu}\frac{\delta I}{\delta\phi_{,\nu}} \tag{31}$$

are the Euler-Lagrange equations of I and the field W^μ is defined by

$$W^\mu = -\frac{\delta I}{\delta \phi_{,\mu}} \omega_\phi + \omega_i \left(\frac{\delta I}{\delta \phi_{,\mu}} \phi_{,i} - \delta^{\mu,i} I \right) \tag{32}$$

When eq. (30) is evaluated at the solution $\phi^\star$ of the Euler-Lagrange equation $[I] = 0$ then W^μ must be divergence-free. The form of the divergence free vector field W^μ dictates the form of the geometry of the level-sets of $\phi^\star$. We will now show an example where knowledge of the symmetry and thus the divergence-free W^μ fields makes basic assumptions on the solution space of the corresponding Euler-Lagrange equations possible.

1.2.3 Kepler's Two Body Problem

Keplers two body problem is the problem of calculating the problem of estimating the trajectory of a body of mass m_e (the earth) which is moving within the vicinity of another body with mass m_s (the sun). According to Newton there exists a gravitational force between the masses coming from the energy $V(r)$ of the gravitational field surrounding the mass m_s at the origin in $\mathbb{R}^3$

$$V(\mathbf{r}_e(t)) = -\frac{m_e \cdot m_s}{r} \qquad r = \|\mathbf{r_e} - \mathbf{r_s}\| \tag{33}$$

The kinetic energy of the mass m_e is $\frac{1}{2} m_e \dot{r}^2$ so that the Lagrangian of the path $\mathbf{r}_e(t)$ is

$$L(\mathbf{r}_e(t)) = \frac{1}{2} m_e \dot{r}_e^2 + \frac{1}{2} m_e \dot{r}_s^2 - V(\mathbf{r}_e(t)) \tag{34}$$

The Euler-Lagrange equations are easily computed

$$\ddot{r}_e + \frac{m_s + m_e}{r^2} = 0 \tag{35}$$

The parameter t is the time parameter of the two body system. The Kepler Lagrangian in eq. (34) exhibits a symmetry under four different one parameter Lie group actions, namely the action of time shift and rotations around the three spacial axis (the group $SO(3) \times \mathbb{R}$)

$$t' = t + \delta t \tag{36}$$

$$\mathbf{r}' = \mathbf{r} + \partial_{\theta_i} \mathbf{r}' \delta\theta_i \quad i = x, y \text{or} z \tag{37}$$

where θ_i are rotation around the x-,y- or z-axis. From Noethers theorem there exist four corresponding conserved quantities:

$$\mathcal{H} = \frac{1}{2} m_e \dot{r}^2 + V(\mathbf{r}_e(t)) \quad \textit{time shift} \tag{38}$$

$$l_x = z\dot{y} - y\dot{z} \quad \textit{Rotation around x-axis} \tag{39}$$

$$l_y = z\dot{x} - x\dot{z} \quad \textit{Rotation around y-axis} \tag{40}$$

$$l_z = x\dot{y} - y\dot{x} \quad \textit{Rotation around z-axis} \tag{41}$$

The conserved quantity $\mathcal{H}$ in eq. (38) is the *Hamiltonian Energy* of the two body system. It constant time and thus manifests that the total energy of the two body system does not disipate away since there are no external forces interacting with the two masses m_e and m_s, that is the two body system is a *closed system*. The vector $\mathbf{l} = (l_x, l_y, l_z)$ (eqs. eq. (39) to eq. (41)) is called the *angular momentum* of the masses m_e and m_s as they rotate around eachother. The solutions to the Euler-Lagrange equations in eq. (35) are elliptic curves in the plane orthorgonal to $\mathbf{l}$. The constancy of $\mathbf{l}$ with respect to the special orthogonal group $SO(3)$ comes the fact the plane is embedded in the eucleadian coordinate space with unit metric, rather some general riemanian space.

2 Total Variation

The earliest attempts to optimization in computer vision all had in common, the use of isotropic priors for the regularization of the unknowns to be estimated. For example one of the earliest attempts for image denoising invloves minimizing the functional ([4])

$$E(u) = \int \left(u - u^0\right)^2 dx + \frac{\lambda}{2} \int |\nabla u|^2 dx \tag{42}$$

The first term in eq eq. (42) is the likelyhood which states the minimizer $u^\star$ must be close in its intensity distribution to the given data u^0. The second term, the prior energy imposes smoothness on the minimizer $u^\star$. Both terms are quadratic in u and thus the Euler-Lagrange equations for $E(u)$ are linear in u making them computationally easy to solve. The problem with the prior $\frac{\lambda}{2} \int |\nabla u|^2 dx$ is that it does not allow the solutions $u^\star$ to have discontinuities. Different approaches for anisotropic priors exist, for instance [5] introduced a quadratic prior

$$E_prior = \int (\nabla u)^T D (\nabla u) \tag{43}$$

The operator D is a local 2×2 symmteric valued matrix with eigenvectors tangential to the level-sets of u^0. This is why D steers the direction of the gradients in eq eq. (43) in tangential direction of the level-sets, and thus also of the discontinuities of u and u^0. The upside is that the prior in eq eq. (43) combined with the likelyhood in eq eq. (58) still lead to Euler-Lagrange equations linear in u. The downside of the prior in eq eq. (43) is that the operator field D must be precomputed on the data u^0, e.g with an eigenvalue analysis of the structure tensor.

In the context of shock-filtering ([6, 7, 8]) it was shown that the functional

$$E_{L_1}(u) = \int |\nabla u| dx \tag{44}$$

has the appealing property that it does not penalize larg disconinuities. However its functional derivative with respect to u is ill conditioned in the case $\nabla u \approx 0$. To alleviate the case, [6] chose the approximative prior

$$E_{L_1 approx}(u) = \int \sqrt{|\nabla u|^2 + \epsilon} dx \tag{45}$$

which is well behaved for $\epsilon > 0$. They were able to achieve good results with relatively sharp preserved discontinuities with data u^0 having low SNRs. Never the less in the limit $\epsilon \to 0$ the Euler-Lagrange equations become more and more computationally instable. A theoretically more well conditioned form of TV is needed which we will outline, following ([9]). To do this we need to explore the functionspace the minimizers of eq eq. (44) might belong to. Smooth functions u_{smooth} are functions for which ∇u exists everywhere, thus they may be minimizers of eq eq. (44). But functions $u_{discont}$ containing discontinuities do *not* have finite L_1 norm of their gradients, $E_{L_1}(u_{discont}) = \infty$ since the gradient $\nabla u_{discont}$ does not exist at the disconinuities. A generalization of eq eq. (44) is possible if one assumes ∇u to be a distribution, more precisely a radon meassure in the space $\mathcal{M}(\Omega)$. If there exists a radon meassure $\mu \in \mathcal{M}(\Omega)$, such that for every $\phi \in \mathcal{C}_0(\Omega)$ with compact domain, the following equality holds

$$\int_\Omega u \cdot \mathrm{Div}\phi dx = -\int \phi d\mu < \infty \tag{46}$$

then μ is called the weak derivative of u and we can identify $\nabla u = \mu$. It is then possible to define the functionspace of bounded variation

$$BV = \{u \in L_1(\Omega) \,|\nabla u \in \mathcal{M}(\Omega)\} \tag{47}$$

Now it is possible to define a norm on BV. By virtue of the Hölder relation there exists a scalar C for which we can determin the upper bound of eq eq. (46)

$$\int_\Omega u \cdot \mathrm{Div}\phi dx \leq C\|\phi\|_\infty \tag{48}$$

The scalar C is the norm of the radon meassure ∇u and is called the total variation of u

$$TV(u) = \sup\left\{\int_\Omega u \cdot \mathrm{Div}\phi dx \,|\quad \|\phi\|_\infty \leq 1\right\} \tag{49}$$

As was discussed in [9] the functions u are geometrically piecewise smooth, meaning there exists a partitioning $\{\Omega_k\}$ of Ω such that $(\nabla u)_{\Omega_k}$ are L_1 integrable. If dl_{mk} is a line segment in the intersection $\Omega_m \cap \Omega_k$ then $TV(u)$ can be written in the form

$$TV(u) = \sum_k \|\nabla u_{\Omega_k}\|_{L_1} + \sum_{k<m} \int_{\Omega_k \cap \Omega_m} |u_k - u_m|\, dl_{km} \tag{50}$$

where u_k the value of u on the portion of $\partial\Omega_k$ which is interfacing with Ω_m and vice versa for u_m. The first term in eq. (50) penalizes the smooth parts of u (the gradients $(\nabla u)_{\Omega_k}$) while the second term penalizes the length of the section $\Omega_m \cap \Omega_k$ while maintaining the values $u_{k,m}$ and thus the *jump* $|u_k - u_m|$. It essentially penalizes the curvature of the line interfaceing with both Ω_k and Ω_m. We will make this point clear in the following section.

2.1 The Mean Curvature of Total Variation

In this section we will discuss the geometrical properties of the TV norm in eq. (49). The subgradient of eq. (49) is equal to the set

$$\partial TV(u) = \left\{-\mathrm{Div}\sigma \,\middle|\quad \sigma \cdot \nu = 0 \text{ on } \partial\Omega,\, \sigma = \frac{\nabla\, u}{|\nabla u|} \text{ if } |\nabla u| \neq 0\right\} \tag{51}$$

This set defines the set of lines $L(v) = TV(u) + \langle \text{Div}\sigma | v - u \rangle$ tangential to TV at a point $u \in BV$. We define a one parameter Lie group $\gamma(t)$, such that its vectorfield X fullfills the condition

$$X \cdot u\left(\Gamma^X(\mathbf{x}_0, t)\right) = 0 \tag{52}$$

then its integral curves $\Gamma^X(t) = (x(t), y(t))$ are the level sets of u. The level sets Γ^X have a curvature κ and the standard formula for κ is

$$\kappa = \frac{1}{\left\|\dot{\Gamma}^X\right\|_{L_1}^3}(\dot{x} \cdot \ddot{y} - \dot{y} \cdot \ddot{x}) \tag{53}$$

If the vector field X is expressed by the coordinate vector $\xi(\mathbf{x}_0)$ then it can be shown κ is a function of the laplacian relative to the coordinate vector $\xi(\mathbf{x}_0)$.

$$\kappa(\mathbf{x}_0) = \frac{\Delta_{\xi\xi} u(\mathbf{x}_0)}{|\nabla u(\mathbf{x}_0)|} \tag{54}$$

This form can easily be transformed into a divergence quantity

$$\kappa = \text{Div}\left(\frac{\nabla\ u}{|\nabla u|}\right) \tag{55}$$

This shows us that the subgradient in eq: eq. (51) is equal to the curvature of the levelsets $\Gamma^X(t)$

$$\kappa = -\partial TV(u) \tag{56}$$

The eq. eq. (56) exposes the capital geometrical property of the TV norm: The TV norm penalizes the curvature of the level-sets of an image. As κ is an invariant of the Lie group $SE(2)$, the group of rotations and translations, TV is also an invariant of that group.

2.2 Image Denoising

Image denoising is the problem of estimating a *clean* image $u^\star$ given a noisy image u^0. The image u^0 is connected to $u^\star$ via

$$u^0 = u^\star + n \quad n \sim \mathcal{D} \tag{57}$$

where $\mathcal{D}$ is some distribution and n is a noise term drawn from $\mathcal{D}$. $u^\star$ is estimated from the familly of functionals

$$F(u) = \frac{1}{q}\int_\Omega |u - u^0|^q dx + \lambda TV(u) \tag{58}$$

The degree q of the data term must be matched to the form of the distribution $\mathcal{D}$. Using the subgradient in eq eq. (51) the Euler-Lagrange equations can be calculated

$$[F](u) = \begin{cases} |u^\star - u^0|^{q-2}\left(u^\star - u^0\right) - \lambda \text{Div}\phi \quad in\Omega \\ \phi \cdot \nu = 0 on \partial\Omega \\ \phi = \frac{\nabla u}{|\nabla u|} \quad |\nabla u| - \text{nearly everywhere} \end{cases} \tag{59}$$

As was discussed before the term $\text{Div}\phi$ is equal to the curvature κ on the level-set of u. Thus the parameter λ controls how strong the curvature of the level-sets are penalized. Never the less the functional eq. (58) still permits jumps in the image $u^\star$.

References

[1] Michael J Black and Paul Anandan, "The robust estimation of multiple motions: Parametric and piecewise-smooth flow fields," *Computer vision and image understanding*, vol. 63, no. 1, pp. 75–104, 1996.

[2] E. Noether, "Invariante variationsprobleme," *Nachrichten Der Koeniglichen Gesellschaft Der Wissenschaften Zu Goettingen*, vol. Mathematische-Physikalische Klasse, pp. 235–257, 1918.

[3] E. Noether, "Invariant variation problems," *Transport Theory and Statistical Problem*, vol. 1 (3), pp. 183–207, 1971.

[4] B. K.P. Horn and B. G. Schunck, "Determining optical flow," *Artificial Intelligence*, vol. 17, issues 1-3, pp. 185–203, 1981.

[5] Hans-Hellmut Nagel and Wilfried Enkelmann, "An investigation of smoothness constraints for the estimation of displacement vector fields from image sequences," *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, , no. 5, pp. 565–593, 1986.

[6] Leonid I Rudin, Stanley Osher, and Emad Fatemi, "Nonlinear total variation based noise removal algorithms," *Physica D: Nonlinear Phenomena*, vol. 60, no. 1, pp. 259–268, 1992.

[7] Stanley Osher and Leonid I Rudin, "Feature-oriented image enhancement using shock filters," *SIAM Journal on Numerical Analysis*, vol. 27, no. 4, pp. 919–940, 1990.

[8] Leonid I Rudin, "Images, numerical analysis of singularities and shock filters," 1987.

[9] Kristian Bredies and Dirk Lorenz, "Mathematische bildverarbeitung," *Vieweg+ Teubner*, vol. 4, no. 6, pp. 12, 2011.